4.4: Game Theory
Solving Games, Reduction by Dominance, and Strictly Determined Games

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Previously, we saw how to find an optimal counter-strategy when we already know the strategy of the other player. Next we will see how to find an "optimal strategy" with no knowledge of the other players moves. However, before proceeding to this point, consider the following.

Exercise 1. Let P be the payoff matrix for "rock, paper, scissors." That is to say

$$P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(a) Suppose player B plays scissors half the time and paper the other half. What is player A's optimal counter-strategy?

(b) Suppose player A uses the optimal counter-strategy from (a). What is player B's optimal counter-strategy?

(c) Suppose player B uses the optimal counter-strategy from (b). What is player A's optimal counter-strategy?

(d) If you were to repeat (b) and (c) repeatedly, always assuming that the other player is using the optimal counter-strategy from the previous stage, would the strategies tend towards a stable answer?

Optimal Strategies

Exercise 1 helps to illustrate an important assumption which is common in many applications of game theory.

Fundamental Principle of Game Theory

Each player tries to use its best possible strategy, and assumes that the other player is doing the same.

Strategies found using this assumption will be referred to as optimal strategies.

Example 1. (**Solving a** 2×2 **Game**) Consider the payoff matrix

$$P = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}.$$

(a) Find the optimal strategy for the row player.

(b) Find the optimal strategy for the column player.

(c) Find the expected payoff of the game assuming both players use their optimal strategies.

Finding the expected payoff, and hence the expected winner, of a game under the fundamental principle of game theory; i.e. when both players use their optimal strategies, is called solving the game. Thus, if you are ever asked to solve a 2×2 game, you are being asked to complete steps (a), (b), and (c) in example 1.

<u>Reduction by Dominance</u>

We have seen how to solve a 2×2 game. The following two strategies will allow us to solve some particular games of larger dimensions. Unfortunately, we will not see how to solve *every* game of larger dimensions. This further study into game theory is one possibility for study once we have met the course requirements.

In some instances, there are certain moves that are worse than other moves regardless of what strategy the opponent uses. We use a procedure known as <u>reduction by dominance</u> to remove these worse moves from play and reduce the payoff matrix to smaller dimensions.

Question 1. Let a_1 and a_2 be two different moves that the row player A can make. When is a_1 not viable (or wise) compared to a_2? Let b_1 and b_2 be two different moves that the column player B can make. When is b_1 not viable (or wise) compared to b_2?

Example 2. Use reduction by dominance to reduce the payoff matrix for RTV and CTV given in example 3 of the previous handout.

	Nature Doc	Symphony	Ballet	Opera
Sitcom	2	1	-2	2
Docudrama	-1	1	-1	2
Reality Show	-2	0	0	1
Movie	3	1	-1	1

After reduction by dominance, we can solve the 2×2 game that is remaining, as done in example 1, to solve this particular 4×4 game.

Strictly Determined Games

A game (of any size) is called strictly determined if the optimal strategies are both pure strategies. We will use a procedure of finding the row minima and column maxima to determine this. If there is an entry of the payoff matrix that is both the row minimum and the column maximum, then we refer to this entry as the saddle point. When there is a saddle point, it determines the optimal pure strategies.

Example 3. Solve the following game:

$$\begin{bmatrix} -4 & -3 & 3 \\ 2 & -1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

A General Strategy for Solving Games

1. Reduce by dominance. This should always be your first step.
2. If you were able to reduce to a 1×1 game, you're done. The optimal strategies are the corresponding pure strategies, as they dominate all the others.
3. Look for a saddle point of the reduced game. If it has one, the game is strictly determined, and the corresponding pure strategies are optimal.
4. If your reduced game is 2×2 and has no saddle point, use the method of example 1 to find the optimal mixed strategies.
5. If your reduced game is larger than 2×2 and has no saddle point, your have to use linear programming to solve it, but this will have to wait until chapter 5.